AUGMENTED SYMMETRICAL AND ASYMMETRICAL FACTORIAL DESIGNS

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I. Introduction

The need of including additional treatments in factorial designs is felt in planning experiments to meet various special requirements of the experimenters. Usually the additional treatments are included as (i) controls, (ii) treatment combinations with doses higher than the experimental ones, (iii) fresh treatments and (iv) repetition of existing treatments. These are introduced to obtain direct responses of treatments, an extra point in the response curve, more information on certain treatments and to ascertain efficiencies of new treatments tried as exploratory measure. At present the analysis of augmented factorial designs is carried out separately for the factorial treatments and the additional treatments and comparisons between the different treatments are made against the pooled error m.s. obtained from the two parts of the design.

Das (1954, 1957) and Giri (1957) have given designs and their analysis which are obtainable by augmenting incomplete block designs. But augmentation of factorial designs which appears to be specially useful, does not seem to have received due importance. Healy (1956) discussed the exact analysis of a 2^3 design in 4 plot blocks with two additional treatments per block when there is complete confounding of the three factor interaction. A general method of exact analysis of symmetrical and asymmetrical designs augmented with additional treatments having various types of confounding is not available in published literature. An attempt has been made in this paper to evolve a technique of exact analysis of different types of augmented designs, with both complete and partial confounding. In this paper, methods of analysis of augmented 2^n, 3^n, 3×2^n, 2×3^n designs in general and augmented 2^5, 3^3, 3^4, 3×2^2, 3×2^3, $3^2 \times 2$, $3^3 \times 2$ designs in particular with different schemes of confounding and their relative efficiencies, have been discussed. An important feature of such designs is that the confounded effects can be recovered. Comparison between the additional treatments on one hand and the factorial treatments on the other becomes also readily available. Further the efficiency of error variance is increased due to the augmentation of the degrees of freedom of error variance in the exact analysis of such designs.

II. Augmented Symmetrical Factorial Designs

1. *Augmented 2^n designs with complete confounding*.—We shall consider 2^n designs in b blocks of size k each in r replications with a-additional treatments per block, confounding completely $(b-1)$ interactions. Let B_{hi} be the total of the i-th block in the h-th replication and $B_{.i}$ the total of r blocks having the same set of treatments as in the i-th block, summed up over all the r replications and the total of additional treatments in $B_{.i}$ by $A_{.i}$.

We now define intra block comparison as

$$B'_{.i} = B_{.i} - \frac{k}{a} A_{.i} \tag{1.1}$$

Then the estimate of each one of the confounded interactions is provided from the constrast.

$$X' = \sum_{i=1}^{b} \lambda_i B'_{.i} \tag{1.2}$$

Where $\Sigma \lambda_i = 0$ and the interaction under consideration determines λ_i's.

It can be shown that

$$E(X') = rbk.x \tag{1.3}$$

where 'x' is the effect of the confounded interaction.

The estimate of the confounded interaction to be denoted by X is given by

$$X = \frac{X'}{rbk}. \tag{1.4}$$

The corresponding S.S. of the confounded interaction X is given by

$$S.S.(X) = \left(1 + \frac{k}{a}\right) \cdot \frac{1}{rbk} \cdot (X')^2. \tag{1.5}$$

The variance for the interaction effect is given by

$$\frac{\left(1+\frac{k}{a}\right)}{rbk}\sigma_e^2$$

while in an unconfounded experiment the variance of the same interaction effect is given by

$$\frac{1}{rbk}\sigma'^2_\epsilon.$$

Thus the relative information on the confounded interaction in an augmented completely confounded design comes as:

$$I(X)=\frac{\sigma'^2_e}{rbk} \div \frac{a+k}{a} : \frac{\sigma_e^2}{rbk} = \frac{a}{a+k} \cdot \frac{\sigma'^2_e}{\sigma_e^2}. \tag{1.6}$$

where

σ_e^2 = Error *m.s.* based on block size of $(k+a)$ plots

σ'^2_e = Error *m.s.* based on block size of k plots containing all the treatments.

2. *Augmented* 2^n *designs with partial confounding.*—We shall now consider the partially confounded design 2^n in b blocks with additional treatments per block in r replications confounding in each replication a different set of $(b-1)$ interactions. Now the estimate of each one of the confounded interaction will be provided from all the replications as a weighted estimate the weights being the inverse of variances of the different estimates obtained from the different replications. Writing X' for the adjusted contrast of the confounded interaction from the replicate in which it is confounded and X for the contrast of the interaction over $(r-1)$ replications where it is not confounded, the estimate of the confounded interaction is provided by $\hat{X}$ where

$$\hat{X}=\frac{a(r-1)X'+(a+k)X}{ar+k} \tag{2.1}$$

as it can be shown that the variances of X' and X are in the ratio of

$$(r-1)a:(ar+k).$$

The estimate of the interaction X is given by

$$X=\frac{k+ra}{(2a+k)(r-1)(bk)}\hat{X}. \tag{2.2}$$

The corresponding $S.S.$ for the confounded interaction X is given by

$$S.S.(X) = \frac{(ar + k)(\hat{X})^2}{(r - 1)\, bk\, (a + k)}\,. \tag{2.3}$$

It can be easily shown that the variance of the effect of each one of the interactions is given by

$$\frac{1}{bk\left(r - \dfrac{k}{a + k}\right)}\,\sigma_e^2,$$

while in an unconfounded experiment the corresponding variance is given by

$$\frac{1}{rbk}\,\sigma'^2_e.$$

Thus the relative information on the confounded interaction in an augmented partially confounded design as compared to the unconfounded experiment comes as

$$I(X) = \left[1 - \frac{1}{r}\,\frac{k}{a + k}\right]\frac{\sigma'^2_e}{\sigma_e^2} = \frac{r - 1}{r} + \frac{1}{r}\left[\frac{a}{a + k}\right]\frac{\sigma'^2_e}{\sigma_e^2} \tag{2.4}$$

This result shows that the information on the partially confounded interaction depends only on block size and the additional number of treatments, and for fixed block size the information increases with the increase of number of replications and the additional treatments per block. We further notice that the contribution to the total information made by the replicate in which the interaction is confounded and is given by the second term in (3.2) decreases inversely with the number of the replications.

The relative information on the confounded interaction in augmented partially confounded designs as compared to partially confounded designs comes out as

$$\left[1 + \frac{1}{r - 1}\cdot\frac{a}{a + k}\right]\frac{\sigma'^2_e}{\sigma_e^2} \tag{2.5}$$

which shows that the information of the augmented partially confounded designs is always greater than that of partially confounded designs.

3. *Efficiency of error* m.s. *in the augmented designs*.—The error *m.s.* in the exact analysis of 2^n augmented design possesses more degrees of freedom than that of the pooled *m.s.* provided from the separate

analysis of the factorial and additional treatments and thus becomes more efficient. It can be shown that the *d.f.* for the error *m.s.* in the case of exact analysis is increased by $(br - 1)$ in the augmented completely confounded designs and by $b(r - 1)$ in the augmented partially confounded designs. The information provided by the *m.s.* of the errors can be compared with the help of the formula given by Fisher (1951) according to which the information based on 'n' *d.f.* for an error *m.s.* is proportional to $(n + 1)/(n + 3)$.

4. *Yield curves in augmented designs.*—In 2^n designs with factors each at non-zero levels and α plots of control per block as additional treatments, the methods given by Fisher (1950) may conveniently be adopted for fitting the yield curve.

Now by virtue of control plots, each factor has three levels with totals based on unequal number of observations. Let the totals of the different levels of a factor be denoted by T_0, T_1, T_2 with frequencies p, q, q respectively.

The second degree polynomial

$$y = A + Bx + Cx_{c_2} \tag{4.1}$$

can be fitted to the data. The values of constants are given by the normal equations:

$$\begin{aligned} S_0A + S_1B + S_2C &= I_0 \\ S_1A + (2S_2 + S_1)B + (3S_3 + 2S_2)C &= L_1 \\ S_2A + (3S_3 + 2S_2)B + (6S_4 + 6S_3 + 2S_2)C &= L_2. \end{aligned} \tag{4.2}$$

Where the unknowns A, B, C are the polynomial values and its first two advancing differences, at the working zero at $x = 1$, and

$$\begin{aligned} S_0 &= p + 2q & L_0 &= T_0 + T_1 + T_2 \\ S_1 &= q - p & L_1 &= T_2 - T_1 \\ S_2 &= p & L_2 &= T_0 \end{aligned} \tag{4.3}$$

The normal equation reduces to

$$\begin{aligned} (p + 2q)A + (q - p)B + pC &= T_0 + T_1 + T_2 \\ (q - p)A + (q + p)B - pC &= T_2 - T_1 \\ pA - pB + pC &= T_0 \end{aligned}$$

Solving these equations the values of the constants come out as

$$A = \frac{T_1}{q}, \quad B = \frac{T_2 - T_1}{q} \quad C = \frac{qT_0 + pT_2 - 2pT_1}{pq}. \tag{4.4}$$

5. *Augmented 2^5 designs with complete confounding.*—As an example we shall first consider the analysis of 2^5 designs in 8 plot blocks with two additional treatments per block in r replications confounding completely the interactions ABC, ADE and $BCDE$.

We now define

$$B'_{.i} = B_{.i} - 4A_{.i}. \tag{5.1}$$

The estimate of the confounded interaction ABC is provided from the adjusted contrast $[ABC]'$, where

$$[ABC]' = -B'_{.1} + B'_{.2} + B'_{.3} - B'_{.4} \tag{5.2}$$

and that it can be shown that

$$E\,[ABC]' = 32r\,(abc) \tag{5.3}$$

where (abc) is the effect of interaction ABC.

Thus the estimate of the interaction ABC is given by

$$ABC = \frac{1}{32r}\,[ABC]'. \tag{5.4}$$

The corresponding $S.S.$ for interaction ABC is given by

$$S.S.\,[ABC] = \frac{\{[ABC]'\}^2}{160r}. \tag{5.5}$$

The variance of the interaction effect (abc) is given by

$$\frac{5}{32} \cdot \frac{\sigma_e^2}{r}$$

as compared to

$$\frac{1}{32} \cdot \frac{\sigma'^2_e}{r}$$

in an unconfounded experiment. Thus the relative information on the interaction ABC is given by

$$I\,(ABC) = \frac{1}{5}\,\frac{\sigma'^2_e}{\sigma_e^2}. \tag{5.6}$$

Similarly the estimates of the interaction ADE, $BCDE$ and their corresponding $S.S.$ can be obtained from their adjusted contrasts with the same precision as

$$\frac{1}{5} \cdot \frac{\sigma'^2_e}{\sigma_e^2}.$$

6. *Augmented 2^5 designs with partial confounding.*—We shall now consider 2^5 augmented partially confounded design in 8 plot blocks

with two additional treatments per block in two replications confounding the interactions *ABC*, *ADE* and *BCDE* in the first replication and the interactions *ABD*, *BCE* and *ACDE* in the second replications.

Now the estimate of the confounded interaction *ABC* will be provided from the two replications as a weighted estimate $(ABC)\hat{}$, weights being equal to the inverse of the variances and is given by

$$[\widehat{ABC}] = \frac{2\,[ABC]' + 10\,[ABC]}{12}. \tag{6.1}$$

The estimate of interaction *ABC* is given by

$$ABC = \frac{1}{32}\,[\widehat{ABC}] \tag{6.2}$$

and its corresponding *S.S.* is given by

$$S.S.\ [ABC] = \frac{3}{80}\{[\widehat{ABC}]\}^2. \tag{6.3}$$

The variance of this interaction effect (*abc*) works out to be

$$\frac{5}{192}\cdot\sigma_e^2$$

as compared to

$$\frac{1}{64}\cdot\sigma'^2_e$$

in an unconfounded experiment. Thus the relative information is given by

$$I\,[ABC]' = \frac{\sigma'^2_e}{64} \div \frac{192}{5}\sigma_e^2 = \frac{3}{5}\frac{\sigma'^2_e}{\sigma_e^2}. \tag{6.4}$$

which is greater than $\frac{1}{2}$ as expected otherwise.

The information in augmented partially confounded designs increases with the increase in the number of the replications. In five replications with balanced confounding of three and four factor interactions the information on each one of the confounded interactions is obtained as

$$I = \left(\frac{4}{5} + \frac{1}{5}\cdot\frac{1}{5}\right)\frac{\sigma'^2_e}{\sigma_e^2} = \frac{21}{25}\frac{\sigma'^2_e}{\sigma_e^2}. \tag{6.5}$$

7. *Augmented* 3^n *designs with complete confounding.*—The analysis of general augmented 3^n factorial designs in k plot blocks with a

additional treatments per blocks replicated r times follows the same lines as that of augmented 2^n designs.

In the case of complete confounding the estimate of each one of the confounded effects is provided from the contrast formed by the three adjusted totals $X'_{.i}$, $i = 0, 1, 2$, where $X'_{.i}$ is defined by

$$X'_{.i} = X_{.i} - \frac{k}{a} A_{.i} \tag{7.1}$$

in which (*i*) $X_{.i}$ denotes the treatment combinations over r-replications specified by the equation

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \ldots + \alpha_n x_n = i \bmod 3$$

where the interaction under consideration determines the α_i's and (*ii*) $A_{.i}$ denotes the sum of all the additional treatments where the i-th group of factorial treatmens occur in the r-replications.

It can be shown that

$$E[X'_{.i}] = 3^{n-1} r (x_i - \bar{a}). \tag{7.2}$$

where x_i is the interaction effect and $\bar{a}$ is the mean effect of the a additional treatments. The estimate of the interaction X is given by

$$X = \frac{1}{3^{n-1} r} \text{dev. } (X'_{.0}, X'_{.1}, X'_{.2}). \tag{7.3}$$

The corresponding $S.S.$ of the confounded interaction X is given by

$$S.S.(X) = \frac{a}{a+k} \frac{1}{3^{n-1} r} \left[\sum_{i=0}^{2} (X'_{.i})^2 - \frac{\left(\sum_{i=0}^{2} X'_{.i} \right)^2}{3} \right]. \tag{7.4}$$

The variance of 'x_i' as defined above works out to be

$$\frac{a+k}{a} \frac{1}{3^{n-1} r} \sigma_e^2$$

as compared to

$$\frac{1}{3^{n-1} r} \sigma'^2_e$$

in an unconfounded experiment. Thus the relative information on the confounded interaction X is given by

$$I(X) = \frac{a}{a+k} \frac{\sigma'^2_e}{\sigma_e^2} \tag{7.5}$$

which is exactly the same as obtained in the case of augmented 2^n design.

8. *Augmented* 3^n *design with partial confounding*.—When the design is subjected to partial confounding, the estimate of the i-th component of the partially confounded interaction will be provided from all the replications as a weighted estimate, $\hat{X}_i$, the weights being equal to the inverse of the variances,

i.e.,

$$\hat{X}_i = \frac{a(r-1)X'_i + (a+k)X_{.i}}{ar+k}. \tag{8.1}$$

where X'_i denotes the adjusted contrasted for the i-th component of the interaction from the replicate in which it is confounded, and $X_{.i}$ the contrast of the i-th component of the interaction over $(r-1)$ replications where it is not confounded as defined earlier.

The estimate of each one of the interaction components each of 2 *d.f.* denoted by X, is given by

$$X = \frac{ar+k}{(r-1)\,3^{n-1}(2a+k)} \text{ dev. } [\hat{X}_0, \hat{X}_1, \hat{X}_2]. \tag{8.2}$$

The corresponding *S.S.* for the interaction X is given by

$$S.S.(X) = \frac{ar+k}{(r-1)\,3^{n-1}(a+k)} \times \left[\sum_{i=0}^{2}(\hat{X}_i)^2 - \frac{\left(\sum_{i=0}^{2}\hat{X}_i\right)^2}{3}\right]. \tag{8.3}$$

The variance of such a weighted interaction effect is given by

$$\frac{1}{3^{n-1}\left(r - \dfrac{k}{a+k}\right)}\sigma_e^2,$$

while in an unconfounded experiment it is given by

$$\frac{1}{3^{n-1}r}\sigma'^2_e.$$

Thus the relative information on this confounded interaction comes as

$$I(X) = \left(1 - \frac{1}{r}\frac{k}{a+k}\right)\frac{\sigma'^2_e}{\sigma_e^2} = \frac{r-1}{r} + \frac{1}{r}\left(\frac{a}{a+k}\right)\frac{\sigma'^2_e}{\sigma_e^2} \tag{8.4}$$

which is the same as in augmented 2^n designs.

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9. *Augmented* 3^3 *design with complete confounding.*—As an example we shall consider the case of 3^3 augmented design in 9 plot blocks with two additional treatments per block in r-replications confounding completely the interaction AB^2C^2.

As usual we define the intra bolck comparison as

$$[AB^2C^2]'_{.i} = [AB^2C^2]_{.i} - \frac{9}{2} A_{.i} \qquad (9.1)$$

where $[AB^2C^2]_{.i}$ denotes the treatment combinations specified by

$$[AB^2C^2]_{.i} = x_1 + 2x_2 + 2x_3 = i \bmod. 3 \quad \text{for} \quad i = 0, 1, 2$$

summed up over all the replications.

It can be shown that

$$E\,[AB^2C^2]'_{.i} = 9r\,(ab^2c^2 - \bar{a}). \qquad (9.2)$$

The estimate of the confounded interaction AB^2C^2 is given by

$$AB^2C^2 = \frac{1}{9r} \text{ dev. } \left[[AB^2C^2]'_{.0},\ [AB^2C^2]'_{.1},\ [AB^2C^2]'_{.2} \right] \qquad (9.3)$$

and its corresponding $S.S.$ is given by

$$S.S.\,(AB^2C^2)$$

$$= \frac{2}{99r} \left[\sum_{i=0}^{2} \{[AB^2C^2]'_{.i}\}^2 - \frac{\left[\sum_{i=0}^{2} [AB^2C^2]'_{.i} \right]^2}{3} \right]. \qquad (9.4)$$

The variance of the interaction effect (ab^2c^2) is given by

$$\frac{11}{18r} \sigma_e^2$$

as compared to

$$\frac{1}{9r} \sigma'^2_e$$

in an unconfounded experiment. Thus the relative information on the interaction AB^2C^2 is given by

$$I\,(AB^2C^2) = \frac{\sigma'^2_e}{9r} \div \frac{\sigma_e}{18r} = \frac{2}{11} \frac{\sigma'^2_e}{\sigma_e^2}. \qquad (9.5)$$

There is a great advantage of such augmented designs when the experiment is replicated only once, as it provides sufficient number of degrees of freedom to the error *m.s.* For example, the degree of freedom for the error *m.s.* in 3^3 design augmented with two treatments replicated once are raised from 6 to 10, making error *m.s.* more efficient.

10. *Augmented* 3^3 *design with partial confounding.*—In the case of augmented partially confounded design of 3^3 in 9 plot blocks in two replications confounding the interaction AB^2C^2 in first replication and AB^2C in the second replication, estimate of each set of the confounded interactions is provided from the two replicates as a weighted estimate the weights being the inverse of variances. For the interaction AB^2C^2, the weighted estimate of its i-th set is given by

$$[\widehat{AB^2C^2}]_i = \frac{2\,[AB^2C^2]'_i + 11\,[AB^2C^2]_i}{13}. \qquad (10.1)$$

The estimate of the interaction AB^2C^2 is given by

$$AB^2C^2 = \frac{1}{9}\text{ dev. }\left[\,[\widehat{AB^2C^2}]_0, [\widehat{AB^2C^2}]_1, [\widehat{AB^2C^2}]_2\right] \qquad (10.2)$$

and its corresponding *S.S.* is given by

$$S.S.\,[AB^2C^2]$$

$$= \frac{13}{99}\left[\sum_{i=0}^{2}\{[\widehat{AB^2C^2}]_i\}^2 - \frac{\left[\sum_{i=0}^{2}[\widehat{AB^2C^2}]_i\right]^2}{3}\right]. \qquad (10.3)$$

It can be shown that the variance of this weighted interaction effect

$(\widehat{ab^2c^2})$ is given by

$$\frac{11}{117}\cdot\sigma_e^{\,2}$$

as compared to

$$\frac{1}{18}\cdot\sigma_e'^{\,2}$$

in an unconfounded experiment. Thus the relative information on the partially confounded interaction AB^2C^2 is given by

$$I(AB^2C^2) = \frac{\sigma_e'^{\,2}}{18} \div \frac{11}{117}\,\sigma_e^{\,2} = \frac{13}{22}\,\frac{\sigma_e'^{\,2}}{\sigma_e^{\,2}}\,. \qquad (10.4)$$

11. *Augmented* 3^4 *design with complete confounding.*—The analysis of an augmented 3^4 design in 9 plot blocks with two additional treat-

ments per block replicated r times is analogous to that of the augmented 3^3 design both for complete and partial confounding.

We shall consider the case where only one replication is used confounding the interactions AB^2C^2, ACD^2, ABD, BC^2D^2.

Now defining as before

$$[AB^2C^2]'_{.i} = [AB^2C^2]_{.i} - \frac{9}{2} A_{.i} \tag{11.1}$$

where $[AB^2C^2]_{.i}$ denotes the treatment combinations specified by the equation,

$$x_1 + 2x_2 + 2x_3 = i \text{ mod. } 3$$

the estimate of this confounded interaction AB^2C^2 is given by

$$AB^2C^2 = \frac{1}{27} \text{dev.} \left[[AB^2C^2]_{.0}'; \; [AB^2C^2)_{.1}', \; [AB^2C^2]_{.2}' \right] \tag{11.2}$$

and its corresponding $S.S.$ is given by

$$S.S.\,(AB^2C^2) = \frac{2}{297} \left[\sum_{i=0}^{2} \{[AB^2C^2]'_{.i}\}^2 - \left[\sum_{i=0}^{2} [AB^2C^2]'_{.i} \right]^2 \right]. \tag{11.3}$$

The variance of the interaction effect (ab^2c^2) is given by

$$\frac{11}{54} \sigma_e^2$$

as compared to

$$\frac{1}{27} \sigma'^2_e$$

in an unconfounded experiment and thus the relative information on the interaction AB^2C^2 is

$$I\,[AB^2C^2] = \frac{2}{11} \frac{\sigma'^2_e}{\sigma_e^2} \cdot$$

The estimates of the interactions ACD^2, ABD, BC^2D^2 are obtained exactly in the same way with the same information.

12. *Augmented* 3^4 *design with partial confounding.*—In partial confounding when another replication confounding ABC, AC^2D, AB^2D^2, BC^2D^2 is added, the estimate of the i-th component of each one of

the partially confounded interaction effects is provided from two replications as a weighted estimate, the weights being the inverse of variances. For interaction AB^2C^2, the wieghted estimate of its i-th component is given by

$$[\widehat{AB^2C^2}]_i = \frac{2\,[AB^2C^2]_i' + 11\,[AB^2C^2]_i}{13} \tag{12.1}$$

where $[AB^2C^2]_i'$ is the adjusted total as defined in (11.1) and $[AB^2C^2]_i$ denotes the total of the twenty-seven treatment combinations specified by the equation,

$$x_1 + 2x_2 + 2x_3 = i \bmod.\ 3.$$

The estimate of the interaction AB^2C^2 is given by

$$AB^2C^2 = \frac{1}{27}\,\text{dev.}\left[[\widehat{AB^2C^2}]_0,\ [\widehat{AB^2C^2}]_1,\ [\widehat{AB^2C^2}]_2\right] \tag{12.2}$$

and the corresponding $S.S.$ of the interaction AB^2C^2 is given by

$$S.S.\,[AB^2C^2] = \frac{13}{297}\left[\sum_{i=0}^{2}\{[\widehat{AB^2C^2}]_i^2\} - \frac{\left[\sum_{i=0}^{2}[\widehat{AB^2C^2}]_i\right]^2}{3}\right]. \tag{12.3}$$

The variance of the confounded interaction effect (ab^2c^2) works out to be

$$\frac{11}{351}\sigma_e^2$$

as compared to

$$\frac{1}{54}\sigma_e'^2$$

in an unconfounded experiment. Thus the relative information on the confounded interaction AB^2C^2 in partially augmented design is given by

$$I\,[AB^2C^2]\,\frac{13}{22}\cdot\frac{\sigma_e'^2}{\sigma_e^2}\,.$$

The estimates for the other partially confounded interactions and their $S.S.$ are obtained exactly in the same manner with the same information.

III. Augmented Asymmetrical Factorial Designs

13. *Augmented* 3×2^n *designs in blocks of* $3 \times 2^{n-1}$ *plots.*—For the present investigation we shall consider designs where the n-factor interaction involving all the factors at two levels and the interaction between this interaction and the factor at three levels are partially confounded. In these designs three replications constituting a balanced design are required for estimating the confounded effects. If the block totals including the additional treatments in the three replications are denoted by B_{11}, B_{12},; B_{21}, B_{22}; B_{31}, B_{32} respectively and that of the differences between block totals belonging to the same replication by

$$B_{12} - B_{11} = G_1; \quad B_{22} - B_{21} = G_2; \quad B_{32} - B_{31} = G_3.$$

and the n-factor interaction at two levels is denoted by X and that of the main effect of the factor at three levels by Y, then the estimate of the interaction X is provided by

$$3(k + a)[X]' = 3(k + a)[X] + k(G_1 + G_2 + G_3) \qquad (13.1)$$

where $[X]$ is the unadjusted contrast for the interaction X.

It can be shown that

$$E\{3(k + a)[X]'\} = N(9a + 8k)(x) \qquad (13.2)$$

where (x) is the effect of the n-factor interaction X and N is the total number of plots in a single replication.

The estimate of the n-factor interaction X is thus given by

$$X = \frac{3(k + a)[X]'}{N(9a + 8k)} \qquad (13.3)$$

and its corresponding $S.S.$ is given by

$$S.S.(X) = \frac{\{3(k + a)[X]'\}^2}{3N(9a + 8k)(k + a)}. \qquad (13.4)$$

The variance of the effect of the interaction X is given by

$$\frac{3(k + a)}{N(9a + 8k)} \sigma_e^2$$

as compared to

$$\frac{1}{Nr} \sigma'^2_e$$

in an unconfounded experiment. Thus the relative information on the n-factor interaction X comes as

$$\mathrm{I}(X) = \left[1 - \frac{k}{9(k+a)}\right] \frac{\sigma'^2_e}{\sigma_e^2}. \tag{13.5}$$

Similarly, the estimate of the interaction between the factor at three levels and the n-factor interaction Y, be denoted by XY, is provided by the following equations:

$$\begin{aligned} 3(k+a) R_0 &= 3(k+a)[X]_{y_0} + k(-G_1 + G_2 + G_3) \\ 3(k+a) R_1 &= 3(k+a)[X]_{y_1} + k(G_1 - G_2 + G_3) \\ 3(k+a) R_2 &= 3(k+a)[X]_{y_2} + k(G_1 + G_2 - G_3) \end{aligned} \tag{13.6}$$

where $[X]_{yi}$ denotes the interaction X as obtained from the i-th level of Y.

It can be shown that

$$E\{3(k+a) R_i\} = \frac{N(5k+9a)}{3}[xy]_i + Nk(x) \tag{13.7}$$

where $[xy]_i$ is the effect of interaction $[X]_{yi}$. The estimate of the interaction XY is given by

$$XY = \frac{3}{N(5k+9a)} \text{dev.} [3(k+a) R_0,\ 3(k+a) R_1,$$
$$3(k+a) R_2]. \tag{13.8}$$

The corresponding $S.S.$ of the interaction XY is given by

$$S.S.(XY) = \frac{\text{dev.}^2 [3(k+a) R_0,\ 3(k+a) R_1,\ 3(k+a) R_2]}{N(5k+9a)(k+a)}. \tag{13.9}$$

The variance of the effect of the interaction XY is given by

$$\frac{9(k+a)}{N(5k+9a)} \sigma_e^2$$

as compared to

$$\frac{1}{N} \sigma'^2_e$$

in an unconfounded experiment. Thus the relative information on the interaction XY comes as

$$I(XY) = \left[1 - \frac{4k}{9(k+\alpha)}\right]\frac{\sigma'^2_e}{\sigma_e^2}. \tag{13.10}$$

14. *Augmented* 3×2^n *design in blocks of* $3\times2^{n-2}$.—In these designs three $(n-1)$-factor interactions of factors at two levels, and the interaction between these and the factor at three levels may be chosen for partial confounding. In these designs also, three replications are required for estimating the confounded effects. If the following differences between the block totals in the first replication are denoted as,

$$G_1 = B_{11} + B_{12} - B_{13} - B_{14}$$

$$G_1' = B_{11} - B_{12} + B_{13} - B_{14}$$

$$G_1'' = B_{11} - B_{12} - B_{13} + B_{14}$$

with similar expressions for replications II and III, the formulæ for the estimates of each one of the partially confounded interactions, their corresponding *S.S.* and information are identical to those of the augmented designs in blocks of $3\times2^{n-1}$, except for the introduction of dashes in G's as defined above relating to various partially confounded interactions.

15. *Augmented* 3×2^2 *design*.—As an example we shall consider a balanced design of 3×2^2 in 6 plot blocks in three replications confounding partially the interactions BC and ABC augmented with α additional treatments per block. Let the block totals including the additional treatments in the three replications be denoted by B_{11}, B_{12}; B_{21}, B_{22}; B_{31}, B_{32}; respectively and that of the differences between block totals by

$$B_{12} - B_{11} = G_1,\ B_{22} - B_{21} = G_2,\ B_{32} - B_{31} = G_3. \tag{15.1}$$

Then we find

$$(6+\alpha)\,Q = (6+\alpha)\,[BC] + 2\,(G_1 + G_2 + G_3) \tag{15.2}$$

where $[BC]$ is the unadjusted total of the interaction BC.

It can be shown that the expected value of the same is given by

$$E\{(6+\alpha)\,Q\} = 12\,(16+3\alpha)\,[bc] \tag{15.3}$$

where $[bc]$ is its interaction effect.

The estimate of the interaction BC is given by

$$BC = \frac{(6+\alpha)\,Q}{12\,(16+3\alpha)}. \tag{15.4}$$

The corresponding $S.S.$ of the interaction BC is given by

$$S.S.\,(BC) = \frac{[(6+\alpha)\,Q]^2}{12\,(6+\alpha)\,(16+3\alpha)}\,. \tag{15.5}$$

The variance of the interaction effect $[bc]$ is given by

$$\frac{(6+\alpha)}{12\,(16+3\alpha)}\,\sigma_e^{\,2}$$

as compared to

$$\frac{1}{36}\,\sigma'^{\,2}_e$$

in an unconfounded experiment. Thus the relative information in an augmented design is given by

$$I\,(BC) = \left[1 - \frac{2}{3\,(6+\alpha)}\right]\frac{\sigma'^{\,2}_e}{\sigma_e^{\,2}}\,. \tag{15.6}$$

The interaction ABC is obtained in a similar manner from the three adjusted totals

$$(6+\alpha)\,R_0 = (6+\alpha)\,[BCa_0] - 2G_1 + 2G_2 + 2G_3$$
$$(6+\alpha)\,R_1 = (6+\alpha)\,[BCa_1] + 2G_1 - 2G_2 + 2G_3$$
$$(6+\alpha)\,R_2 = (6+\alpha)\,[BCa_2] + 2G_1 + 2G_2 - 2G_3 \tag{15.7}$$

where $[BCa_i]$ is the interaction total of BC at i-th level of 'A'. It can be shown that

$$E\,(6+\alpha)\,R_i = 4\,(10+3\alpha)\,[abc]_i + 24\,[bc] \tag{15.8}$$

where $[abc]_i$ is the effect of interaction of $[BC]\,a_i$.

The estimate of the interaction ABC is given by

$$ABC = \frac{1}{4\,(10+3\alpha)}\,\text{dev.}\,[(6+\alpha)\,R_0,\ (6+\alpha)\,R_1,\ (6+\alpha)\,R_2] \tag{15.9}$$

and the corresponding $S.S.$ is given by

$$S.S.\,[ABC] = \frac{\text{dev.}^2\,[(6+\alpha)\,R_0,\ (6+\alpha)\,R_1,\ (6+\alpha)\,R_2]}{4\,(6+\alpha)\,(10+3\alpha)}\,. \tag{15.10}$$

The variance of the interaction effect $[abc]_i$ is given by

$$\frac{(6+\alpha)}{4\,(10+3\alpha)\,\sigma_e^{\,2}}$$

as compared to

$$\frac{1}{12}\sigma'^2_e$$

in an unconfounded experiment. Thus the relative information on the interaction ABC is given by

$$I(ABC) = \left[1 - \frac{8}{3(6+\alpha)}\right]\frac{\sigma'^2_e}{\sigma_e^2}. \qquad (15.11)$$

16. *Augmented* 3×2^3 *design.*—The analysis of 3×2^3 design in 6 plot blocks in three replications with α additional treatments per block confounding partially the interactions CD, BD, BC, ACD, ABD and ABC is the simple extension of augmented 3×2^2 design. Let the differences between the block totals in replication I be denoted by G_1, G_1', G_1'', where

$$\begin{aligned} G_1 &= B_{11} + B_{12} - B_{13} - B_{14} \\ G_1' &= B_{11} - B_{12} + B_{13} - B_{14} \\ G_1'' &= B_{11} - B_{12} - B_{13} + B_{14} \end{aligned} \qquad (16.1)$$

with similar expressions for replication II and III.

Then we calculate

$$\begin{aligned} (6+\alpha)Q &= (6+\alpha)[CD] + 2(G_1 + G_2 + G_3) \\ (6+\alpha)Q' &= (6+\alpha)[BD] + 2(G_1' + G_2' + G_3') \\ (6+\alpha)Q'' &= (6+\alpha)[BC] + 2(G_1'' + G_2'' + G_3''). \end{aligned} \qquad (16.2)$$

It can be shown that

$$E\{(6+\alpha)Q\} = 24(16+3\alpha)t \qquad (16.3)$$

where t is the effect of interaction CD

with similar expressions for the interactions BD and BC.

The estimates of the interactions CD, BD and BC are given by

$$CD = \frac{(6+\alpha)Q}{24(16+3\alpha)}; \quad BD = \frac{(6+\alpha)Q'}{24(16+3\alpha)};$$

$$BC = \frac{(6+\alpha)Q''}{24(16+3\alpha)}. \qquad (16.4)$$

The corresponding *S.S.* of these interactions are given by

$$S.S.\ [CD] = \frac{\{(6+\alpha)\,Q\}^2}{24\,(6+\alpha)\,(16+3\alpha)}\ ;$$

$$S.S.\ [BD] = \frac{[(6+\alpha)\,Q']^2}{24\,(6+\alpha)\,(16+3\alpha)}\ ;$$

$$S.S.\ [BC] = \frac{\{(6+\alpha)\,Q''\}^2}{24\,(6+\alpha)\,(16+3\alpha)}\,. \tag{16.5}$$

The variance of each one of the interaction effect is given by

$$\frac{(6+\alpha)}{24\,(16+3\alpha)}\,\sigma_e^{\,2}$$

as compared to

$$\frac{1}{72}\,\sigma'^{\,2}_e.$$

Thus the relative information on each one of the two factor interactions is given by

$$\left[1 - \frac{2}{3\,(6+\alpha)}\right]\frac{\sigma'^{\,2}_e}{\sigma_e^{\,2}}\,. \tag{16.6}$$

Further, we find

$$\begin{aligned}(6+\alpha)\,R_0 &= (6+\alpha)\,[CDa_0] - 2G_1 + 2G_2 + 2G_3\\ (6+\alpha)\,R_1 &= (6+\alpha)\,[CDa_1] + 2G_1 - 2G_2 + 2G_3\\ (6+\alpha)\,R_2 &= (6+\alpha)\,[CDa_2] + 2G_1 + 2G_2 - 2G_3.\end{aligned} \tag{16.7}$$

It can be shown that

$$E\,(6+\alpha)\,R_i = 8\,(10+3\alpha)\,[acd]_i + 48\,[cd] \tag{16.8}$$

and therefore the estimate of the interaction *ACD* is given by

$$ACD = \frac{1}{8\,(10+3\alpha)}\,\text{dev.}\ [(6+\alpha)\,R_0,\ (6+\alpha)\,R_1,\ (6+\alpha)\,R_2]\,. \tag{16.9}$$

and the corresponding *S.S.* is given by

$$S.S.\ [ACD] = \frac{\text{dev.}^2\ [(6+\alpha)\,R_0,\ (6+\alpha)\,R_1,\ (6+\alpha)\,R_2]}{8\,(6+\alpha)\,(10+3\alpha)}\,. \tag{16.10}$$

The variance of the interaction effect $[acd]_i$ is given by

$$\frac{(6+a)}{8(10+3a)}\sigma_e^2$$

as compared to

$$\frac{1}{24}\sigma'^2_e$$

in an unconfounded experiment. Thus the relative information on interaction ACD is given by

$$I(ACD) = \left[1 - \frac{8}{3(6+a)}\right]\frac{\sigma'^2_e}{\sigma_e^2}. \qquad (16.11)$$

The other interaction ABD and ABC are obtained in similar manner by introducing single and double dashes in the above formulæ.

17. *Augmented* $3^n \times 2$ *in* $3^{n-1} \times 2$ *plot blocks.*—In these designs a component of the interaction involving the n factors each at three levels and the interaction between this and the factor at two levels are partially confounded. Minimum of two replications constituting a balanced design are required for estimation of the confounded effects. If the component of n-factor partially confounded interaction be denoted by X and that of the factor at two levels by Y, then the estimate of the partially confounded n-factor interaction, X, is provided by

$$2(k+a)Q_0 = 2(k+a)[X_0] - k(B_{11} + B_{12} + B_{22} + B_{23})$$
$$2(k+a)Q_1 = 2(k+a)[X_1] - k(B_{11} + B_{13} + B_{21} + B_{23})$$
$$2(k+a)Q_2 = 2(k+a)[X_2] - k(B_{11} + B_{12} + B_{21} + B_{22}). \qquad (17.1)$$

It can be shown that

$$E\{2(k+a)Q_i\} = k(3k+4a)x_i - k^2\sum^2 x_i - 4k\Sigma a_i \qquad (17.2)$$

where x_i is the effect of the i-th component of the partially confounded interaction X_i and $\sum_i^a a_i$ is the sum of the effects of all the additional treatments.

The estimate of the partially confounded n-factor interaction X is given by

$$X = \frac{1}{k(3k+4a)}\,\text{dev.}\,[2(k+a)Q_0, 2(k+a)Q_1, 2(k+a)Q_2]. \qquad (17.3)$$

The corresponding $S.S.$ of the partially confounded interaction X is given by

$$S.S.(X) = \frac{\text{dev.}^2\,[2(k+a)\,Q_0,\ 2(k+a)\,Q_1,\ 2(k+a)\,Q_2]}{2k(k+a)(3k+4a)}. \tag{17.4}$$

The variance of the effect of partially confounded interaction X is given by

$$\frac{2(k+a)}{k(3k+4a)} \cdot \sigma_e^2$$

as compared to

$$\frac{1}{2k} \cdot \sigma_e'^2$$

in an unconfounded experiment. Thus, the relative informations on partially confounded interaction X is given by

$$I(X) = \left[1 - \frac{k}{4(k+a)}\right] \frac{\sigma_e'^2}{\sigma_e^2}. \tag{17.5}$$

Similarly, the estimate of interaction between n-factor interaction X and the factor at two levels, Y to be denoted by XY, is obtainable from

$$\begin{aligned} 2(k+a)\,R_0 &= 2(k+a)\,[X_0]_{(y_1-y_0)} - k\,(B_{12}-B_{13}-B_{22}+B_{23}) \\ 2(k+a)\,R_1 &= 2(k+a)\,[X_1]_{(y_1-y_0)} - k\,(-B_{11}+B_{13}+B_{21}-B_{23}) \\ 2(k+a)\,R_2 &= 2(k+a)\,[X_2]_{(y_1-y_0)} - k\,(B_{11}-B_{12}-B_{21}+B_{22}). \end{aligned} \tag{17.6}$$

It can be shown that

$$E\{2(k+a)\,R_i\} = k(k+4a)\,[xy]_i + Nk\,[y]. \tag{17.7}$$

The estimate of the partially confounded interaction XY is given by

$$XY = \frac{1}{k(k+4a)}\,\text{dev.}\,[2(k+a)\,R_0,\ 2(k+a)\,R_1,\ 2(k+a)\,R_2]. \tag{17.8}$$

The corresponding $S.S.$ for the partially confounded interaction XY is given by

$$S.S.(XY) = \frac{1}{2k(k+a)(k+4a)} \text{dev.}^2 [2(k+a) R_0,$$

$$2(k+a) R_1,\ 2(k+a) R_2]. \qquad (17.9)$$

The variance of the effect of the partially confounded interaction XY is given by

$$\frac{2(k+a)}{k(k+4a)} \sigma_e^2$$

as compared to

$$\frac{1}{2k} \sigma'^2_e$$

in an unconfounded experiment. Thus, the relative information, on partially confounded interaction XY is given by

$$I(XY) = \frac{k+4a}{2(k+a)} \frac{\sigma'^2_e}{\sigma_e^2} = \left[1 - \frac{3k}{4(k+a)}\right] \frac{\sigma'^2_e}{\sigma_e^2}. \qquad (17.10)$$

18. *Augmented* $3^2 \times 2$ *design.*—As an example we shall consider the augmented design $3^2 \times 2$ in 6 plot blocks in two replications with a additional treatments per block confounding partially AB^2 and AB^2C. Let the block totals in two replications including the additional treatments be denoted by B_{11}, B_{12}, B_{13} and B_{21}, B_{22}, B_{23}. The estimate of the interaction, AB^2, is provided by the three adjusted totals:

$$(6+a)[AB^2]'_0 = (6+a)[AB^2]_{.0} - 3(B_{12} + B_{13} + B_{22} + B_{23})$$

$$(6+a)[AB^2]'_1 = (6+a)[AB^2]_{.1} - 3(B_{11} + B_{13} + B_{21} + B_{23})$$

$$(6+a)[AB^2]'_2 = (6+a)[AB^2]_{.2} - 3(B_{11} + B_{12} + B_{21} + B_{22})$$

(18.1)

in which $[AB^2]_{.i}$ is the total of the treatment combinations specified by the equation $x_1 + 2x_2 = i$ mod. 3, for $i = 0, 1, 2$ summed up over all the replications.

It can be shown that

$$E(6+a)[AB^2]_i' = 6(9+2a)[ab^2]_i - 18 \sum_i [ab^2]_i - 12 \sum_i^a a_i.$$

(18.2)

The estimate of interaction AB^2 is given by

$$AB^2 = \frac{1}{6(9+2a)} \text{ dev.} \Big[(6+a)\,[AB^2]'_0,\ (6+a)\,[AB^2]'_1,$$

$$(6+a)\,[AB^2]'_2 \Big]. \tag{18.3}$$

The corresponding $S.S.$ of the interaction AB^2 is given by

$$S.S.\,(AB^2) = \frac{1}{6(6+a)(9+2a)} \text{ dev.}^2 \Big[(6+a)\,[AB^2]'_0,$$

$$(6+a)\,[AB^2]'_1,\ (6+a)\,[AB^2]'_2 \Big]. \tag{18.4}$$

The variance of the interaction effect $[ab^2]_i$ is given by

$$\frac{(6+a)}{6(9+2a)}\sigma_e^2$$

as compared to

$$\frac{1}{12}\sigma'^2_e$$

in an unconfounded experiment. Thus the relative information on the partially confounded interaction AB^2 is given by

$$I_1\,(AB^2) = \left[1 - \frac{3}{2(6+a)}\right] \frac{\sigma'^2_e}{\sigma_e^2}. \tag{18.5}$$

By obtaining R_0, R_1 and R_2 as below:

$$(6+a)\,R_0 = (6+a)\,[AB^2]_{.0(c_1-c_0)} - 3\,(B_{12} + B_{23} - B_{13} - B_{22})$$

$$(6+a)\,R_1 = (6+a)\,[AB^2]_{.1(c_1-c_0)} - 3\,(B_{13} + B_{21} - B_{11} - B_{23})$$

$$(6+a)\,R_2 = (6+a)\,[AB^2]_{.2(c_1-c_0)} - 3\,(B_{11} + B_{22} - B_{12} - B_{21})$$

(18.6)

it can be shown that

$$E\{(6+a)\,R_i\} = 6\,(3+2a)\,[ab^2c]_i \tag{18.7}$$

The estimate of the interaction AB^2C is given by

$$AB^2C = \frac{1}{6(3+2a)} \text{ dev.}\,[(6+a)\,R_0,\ (6+a)\,R_1,\ (6+a)\,R_2]$$

(18.8)

and the corresponding *S.S.* is given by

$$S.S.\,(AB^2C) = \frac{1}{6\,(6+\alpha)\,(3+2\alpha)}\,\text{dev.}^2\,[(6+\alpha)\,R_0,\;(6+\alpha)\,R_1,\;(6+\alpha)\,R_2]. \tag{18.9}$$

The variance of the interaction effect $[ab^2c]_i$ is given by

$$\frac{(6+\alpha)}{6\,(3+2\alpha)}\,\sigma_e^2$$

as compared to

$$\frac{1}{12}\,\sigma'^2_e$$

in an unconfounded experiment.

Thus, the relative information is given by

$$I_1\,(AB^2C) = \left[1 - \frac{9}{2\,(6+\alpha)}\right]\frac{\sigma'^2_e}{\sigma_e^2}. \tag{18.10}$$

When the design is supplemented by two more replications confounding partially *AB* and *ABC*, the estimate of each one of the partially confounded interactions of factors at three levels is then provided from the two estimate obtainable from the two designs each consisting of two replications, as a weighted estimate, the weights being the inverse of the variances of the individual estimate.

The weighted estimate of the *i*-th component of partially confounded interaction AB^2 will be provided by $[\widehat{AB^2}]_i$ where

$$[\widehat{AB^2}]_i = \frac{2\,(6+\alpha)\,(AB^2)'_{.i} + (6+\alpha)\,(9+2\alpha)\,[AB^2]_{.i}}{(6+\alpha)\,(9+2\alpha)+2}. \tag{18.11}$$

The corresponding *S.S.* for the partially confounded interaction AB^2 is given by

$$S.S.\,(AB^2) = \frac{(6+\alpha)\,(9+2\alpha)+2}{12\,(6+\alpha)\,(9+2\alpha)}\,\text{dev.}^2\,\{[\widehat{AB^2}]_0,\;[\widehat{AB^2}]_1,\;[\widehat{AB^2}]_2\}. \tag{18.12}$$

The variance of the interaction effect $[ab^2]_i$ is given by

$$\frac{(6+\alpha)}{6\,(21+4\alpha)}\,\sigma_e^2$$

as compared to

$$\frac{1}{24}\sigma'^2_e$$

in an unconfounded experiment. Thus the relative information on the partially confounded interaction AB^2 in a balanced design comes as

$$I_2(AB^2) = \left[1 - \frac{3}{4(6+\alpha)}\right]\frac{\sigma_e'^2}{\sigma_e^2}. \qquad (18.13)$$

Similarly, the weighted estimate of i-th component of the partially confounded interaction AB^2C is given by

$$[\widehat{AB^2C}]_i = \frac{2(6+\alpha)[AB^2C]'_i + (6+\alpha)(3+2\alpha)[AB^2C]_i}{(6+\alpha)(3+2\alpha)+2}. \qquad (18.14)$$

The corresponding $S.S.$ for the partially confounded interaction AB^2C is given by

$$S.S.(AB^2C) = \frac{(6+\alpha)(3+2\alpha)+2}{12(6+\alpha)(3+2\alpha)}\text{dev.}^2\{[\widehat{AB^2C}]_0, [\widehat{AB^2C}]_1, [\widehat{AB^2C}]_2\}. \qquad (18.15)$$

The variance of the partially confounded interaction effect $[ab^2c]_i$ is given by

$$\frac{(6+\alpha)}{6(15+2\alpha)}\sigma_e^2$$

as compared to

$$\frac{1}{24}\sigma'^2_e$$

in an unconfounded experiment. Thus the relative information on the partially confounded interaction ABC in a balanced design with two sets of two replications each is given by

$$I_2(AB^2C) = \left[1 - \frac{9}{4(6+\alpha)}\right]\frac{\sigma'^2_e}{\sigma_e^2}, \qquad (18.16)$$

which is the same as

$$I_2(AB^2C) = \frac{S_n}{S} + \frac{S_p}{S}I_1(AB^2C) \qquad (18.17)$$

where

S_p = Number of sets in which the interaction is partially confounded.

S_n = Number of sets in which the interaction is not confounded.

$S = S_p + S_n$ = Total number of sets of two replications each used in the design.

I_1 = Information of interaction from a single set of design where it is confounded.

The weighted estimates on partially confounded interactions AB and ABC, their $S.S.$ and information are exactly identical to that of the partially confounded interactions AB^2 and AB^2C

19. *Augmented* $3^n \times 2$ *design in* $3^{n-2} \times 2$ *plot blocks*.—In these designs components of three $(n-1)$-factor interactions of factors at three levels, interactions of these three interactions with the factor at two levels are taken for partially confounding and component of n-factor interaction of factors at three levels is taken for complete confounding.

The estimates of these six partially confounded interactions and one completely confounded interaction and their corresponding $S.S.$ are obtained in the same way as in augmented $3^n \times 2$ design in $3^{n-1} \times 2$ plot blocks.

20. *Augmented* $3^3 \times 2$ *design in* 6 *plot blocks*.—As an example we shall consider an augmented design $3^3 \times 2$ with a additional treatments in a set of two replications confounding partially interactions AB^2, AC^2, BC^2 and AB^2D, AC^2D, BC^2D and completely interaction ABC. Let us denote the block totals including the additional treatments in both the replications by $B_{11}, B_{12}, \ldots, B_{17}$; and $B_{21}, B_{22}, \ldots, B_{19}$ respectively.

The estimates of the partially confounded interaction AB^2 is provided from the contrast formed by the following three quantities:

$$(6 + a)\,[AB^2]'_{.0} = (6 + a)\,[AB^2]_{.0} - 3\,(B_{11} + B_{13} + B_{15} + B_{16} + B_{17} + B_{18} + B_{21} + B_{23} + B_{25} + B_{26} + B_{27} + B_{28})$$

$$(6 + a) [AB^2]'_{.1} = (6 + a) [AB^2]_{.1} - 3 (B_{11} + B_{12} + B_{14} + B_{16} + B_{18} + B_{19} + B_{21} + B_{22} + B_{24} + B_{26} + B_{28} + B_{29})$$

$$(6 + a) [AB^2]'_{.2} = (6 + a) [AB^2]_{.2} - 3 (B_{12} + B_{13} + B_{14} + B_{15} + B_{17} + B_{19} + B_{22} + B_{23} + B_{24} + B_{25} + B_{27} + B_{29}). \quad (20.1)$$

It can be shown that

$$E\{(6 + a) [AB^2]'_{.i}\} = 18 (9 + 2a) [ab^2]_i - 54 \sum_i [ab]_i^2 - 36 \sum^{a} a_i \quad (20.2)$$

where a_i is the effect of the i-th additional treatment.

The estimate of the partially confounded interaction AB^2 is given by

$$AB^2 = \frac{1}{18 (9 + 2a)} \text{dev.} \{(6 + a) [AB^2]'_{.0}, (6 + a) [AB^2]'_{.1}, (6 + a) [AB^2]'_{.2}\}. \quad (20.3)$$

The corresponding $S.S.$ of the partially confounded interaction AB^2 is given by

$$S.S. (AB^2) = \frac{1}{18 (9 + 2a) (6 + a)} \text{dev.}^2 \{(6 + a) [AB^2]'_{.0}, (6 + a) [AB^2]'_{.1}, (6 + a) [AB^2]'_{.2}\}. \quad (20.4)$$

The variance of the effect of the partially confounded interaction AB^2 is given by

$$\frac{(6 + a)}{18 (3 + 2a)} \sigma_e^2$$

as compared to

$$\frac{1}{36} \sigma'^2_e$$

in an unconfounded experiment. Thus the relative information is given by

$$I_1 (AB^2) = \left[1 - \frac{3}{2 (6 + a)}\right] \frac{\sigma'^2_e}{\sigma_e^2}. \quad (20.5)$$

The expressions for the estimates of the partially confounded interactions AC^2, BC^2, are obtained exactly in the same manner as that of the partially confounded interaction AB^2, with the same information as given in (21.5).

For estimation of AB^2D we calculate

$$(6+\alpha)\,[AB^2D]'_{.0} = [AB^2D]_{.0} - 3\,(B_{13}+B_{15}+B_{17}+B_{21}+B_{26}+B_{28} - B_{11}-B_{16}-B_{18}-B_{23}-B_{25}-B_{27})$$

$$(6+\alpha)\,[AB^2D]'_{.1} = [AB^2D]_{.1} - 3\,(B_{11}+B_{16}+B_{18}+B_{22}+B_{24}+B_{29} - B_{12}-B_{14}-B_{19}-B_{21}-B_{26}-B_{28})$$

$$(6+\alpha)\,[AB^2D]'_{.2} = [AB^2D]_{.2} - 3\,(B_{12}+B_{14}+B_{19}+B_{23}+B_{25}+B_{27} - B_{13}-B_{15}-B_{17}-B_{22}-B_{24}-B_{29}). \qquad (20.6)$$

It can be shown that

$$E\,(6+\alpha)\,[AB^2D]'_{.i} = 18\,(3+2\alpha)\,[ab^2d]_i + 54 \sum_i [ab^2d]_i. \qquad (20.7)$$

The estimate of AB^2D is given by

$$AB^2D = \frac{1}{18\,(3+2\alpha)}\ \text{dev.}\,\{(6+\alpha)\,[AB^2D]'_{.0},\ (6+\alpha)\,[AB^2D]'_{.1},\ (6+\alpha)\,[AB^2D]'_{.2}\}. \qquad (20.8)$$

The corresponding $S.S.$ of the partially confounded interaction AB^2D is given by

$$S.S.\,(AB^2D) = \frac{1}{18\,(6+\alpha)\,(3+2\alpha)}\ \text{dev.}^2\,\{(6+\alpha)\,[AB^2D]'_{.0},\ (6+\alpha)\,[AB^2D]'_{.1},\ (6+\alpha)\,[AB^2D]'_{.2}\}. \qquad (20.9)$$

The variance of effect of the partially confounded interaction AB^2D is given by

$$\frac{(6+\alpha)}{18\,(3+2\alpha)}\,\sigma_e^{\,2}$$

as compared to

$$\frac{1}{36}\sigma'^2_e$$

in an unconfounded experiment. Thus the relative information on the partially confounded interaction ABD comes as

$$I_1(AB^2D) = \left[1 - \frac{9}{2(6+\alpha)}\right]\frac{\sigma'^2_e}{\sigma^2_e}. \tag{20.10}$$

Estimates of other partially confounded interaction AC^2D, BC^2D and their $S.S.$ are founded exactly in the same manner as above with information as given in (21.10).

Further we calculate

$$(6+\alpha)[ABC]'_{.0} = (6+\alpha)[ABC]_{.0} - 6(B_{11}+B_{12}+B_{13}+B_{21}+B_{22}+B_{23})$$

$$(6+\alpha)[ABC]'_{.1} = (6+\alpha)[ABC]_{.1} - 6(B_{14}+B_{15}+B_{16}+B_{24}+B_{25}+B_{26})$$

$$(6+\alpha)[ABC]_{.2} = (6+\alpha)[ABC]_{.2} - 6(B_{17}+B_{18}+B_{19}+B_{27}+B_{28}+B_{29}). \tag{20.11}$$

It can be shown that

$$E(6+\alpha)[ABC]'_{.i} = 36\alpha[abc]_i - 36\sum^{\alpha} a_i. \tag{20.12}$$

The estimate of the completely confounded interaction ABC is given by

$$ABC = \frac{1}{36\alpha}\text{dev.}\{(6+\alpha)[ABC]'_{.0}, (6+\alpha)[ABC]'_{.1}, (6+\alpha)[ABC]'_{.2}\}. \tag{20.13}$$

The corresponding $S.S.$ of completely confounded interaction ABC is given by

$$S.S.(ABC) = \frac{1}{36(6+\alpha)\alpha}\text{dev.}^2\{(6+\alpha)[ABC]'_{.0}, (6+\alpha)[ABC]'_{.1}, (6+\alpha)[ABC]'_{.2}\}. \tag{20.14}$$

The variance of the effect of completely confounded interaction *ABC* is given by

$$\frac{(6+\alpha)}{36\alpha}\sigma_e^2$$

as compared to

$$\frac{1}{36}\sigma'^2_e$$

in an unconfounded experiment. Thus the relative information on the completely confounded interaction *ABC* is given by

$$I_1(ABC) = \frac{\alpha}{6+\alpha} \cdot \frac{\sigma_e'^2}{\sigma_e^2}. \tag{20.15}$$

When all four sets with different system of confounding are used to make the design balanced, the estimate of each one of partially and completely confounded interactions is obtained from the four sets, each consisting of two replications, as a weighted estimate, weights, being the inverse of variances, in the same manner as done for balanced $3^2 \times 2$ design.

It can be shown that the information on different partially and completely confounded interactions are as below:

Information on the partially confounded interactions of factors at three levels

$$= \left[1 - \frac{3}{4(6+\alpha)}\right]\frac{\sigma'^2_e}{\sigma_e^2}.$$

Information on interaction between the partially confounded interaction of factors at three levels and the factor at two levels

$$= \left[1 - \frac{9}{4(6+\alpha)}\right]\frac{\sigma'^2_e}{\sigma_e^2}.$$

Information on completely confounded interaction:

$$= \left[1 - \frac{3}{2(6+\alpha)}\right]\frac{\sigma'^2_e}{\sigma_e^2}.$$

The results are derived from the general formula

$$I_2 = \frac{S_n}{S} + \frac{S_g}{S}I_1$$

as given before.

IV. SUMMARY

In this paper a general method for exact analysis of symmetrical and asymmetrical Factorial Designs augmented with additional treatments for various schemes of confounding and their relative efficiencies have been discussed. In particular the analyses of 2^5, 3^3, 3^4, 3×2^2, 3×2^3, $3^2\times2$, $3^3\times2$ designs have been worked out. Further, it has been shown that in such confounded designs, all effects are recoverable and that the efficiency of error variance is increased due to augmentation of degrees of freedom in the exact analyses. A method of fitting a response curve in augmented designs, especially in augmented 2^n designs, has also been given, where augmentation provides one additional point.

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